CAMBRIDGE ASSESSMENT

Sixth Term Examination Papers **9470**

MATHEMATICS 2 Afternoon

MONDAY 20 JUNE 2011 Time: 3 hours

3201393311

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 10 printed pages and 2 blank pages.

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Section A: Pure Mathematics

1 **(i)** Sketch the curve $y = \sqrt{1-x} + \sqrt{3+x}$.

Use your sketch to show that only one real value of x satisfies

$$\sqrt{1-x} + \sqrt{3+x} = x + 1\,,$$

and give this value.

(ii) Determine graphically the number of real values of x that satisfy

$$2\sqrt{1-x} = \sqrt{3+x} + \sqrt{3-x}\,.$$

Solve this equation.

2 Write down the cubes of the integers 1, 2, ... , 10 .

The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3\,, \qquad (*)$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that

$$z^3 = k^2 - 3kx + 3x^2\,.$$

Deduce that $(4z^3 - k^2)/3$ is a perfect square and that $\frac{1}{4}k^2 \leqslant z^3 < k^2$.

Use these results to find a solution of $(*)$ when $k = 20$.

(ii) By considering the case $x + y = z^2$, find two solutions of $(*)$ when $k = 19$.

3 In this question, you may assume without proof that any function f for which $f'(x) \geqslant 0$ is *increasing*; that is, $f(x_2) \geqslant f(x_1)$ if $x_2 \geqslant x_1$.

(i) **(a)** Let $f(x) = \sin x - x\cos x$. Show that $f(x)$ is increasing for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and deduce that $f(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(b) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) \geqslant 1$ for $0 \leqslant x < 1$, show that

$$\arcsin x \geqslant x \qquad (0 \leqslant x < 1).$$

(c) Let $g(x) = x \operatorname{cosec} x$ for $0 < x < \frac{1}{2}\pi$. Show that g is increasing and deduce that

$$(\arcsin x)\, x^{-1} \geqslant x \operatorname{cosec} x \qquad (0 < x < 1).$$

(ii) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arctan x) \leqslant 1$ for $x \geqslant 0$, show by considering the function $x^{-1}\tan x$ that

$$(\tan x)(\arctan x) \geqslant x^2 \qquad (0 < x < \tfrac{1}{2}\pi).$$

4 **(i)** Find all the values of θ, in the range $0° < \theta < 180°$, for which $\cos\theta = \sin 4\theta$. Hence show that

$$\sin 18° = \frac{1}{4}\left(\sqrt{5} - 1\right).$$

(ii) Given that

$$4\sin^2 x + 1 = 4\sin^2 2x,$$

find all possible values of $\sin x$, giving your answers in the form $p + q\sqrt{5}$ where p and q are rational numbers.

(iii) Hence find two values of α with $0° < \alpha < 90°$ for which

$$\sin^2 3\alpha + \sin^2 5\alpha = \sin^2 6\alpha.$$

5 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A. Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$$

where $\lambda = \dfrac{2\,\mathbf{a}.\mathbf{b}}{\mathbf{a}.\mathbf{a}}$.

The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d} = \mu\mathbf{b} - \lambda\mathbf{a}$$

for some scalar μ to be determined.

Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda = -\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

6 For any given function f, let

$$I = \int [\mathrm{f}'(x)]^2\,[\mathrm{f}(x)]^n \mathrm{d}x\,, \qquad (*)$$

where n is a positive integer. Show that, if $\mathrm{f}(x)$ satisfies $\mathrm{f}''(x) = k\mathrm{f}(x)\mathrm{f}'(x)$ for some constant k, then $(*)$ can be integrated to obtain an expression for I in terms of $\mathrm{f}(x)$, $\mathrm{f}'(x)$, k and n.

(i) Verify your result in the case $\mathrm{f}(x) = \tan x$. Hence find

$$\int \frac{\sin^4 x}{\cos^8 x}\,\mathrm{d}x\,.$$

(ii) Find

$$\int \sec^2 x\,(\sec x + \tan x)^6\,\mathrm{d}x\,.$$

7 The two sequences a_0, a_1, a_2, ... and b_0, b_1, b_2, ... have general terms

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n\,,$$

respectively, where $\lambda = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$.

(i) Show that $\displaystyle\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}}\, a_{n+1}$, and give a corresponding result for $\displaystyle\sum_{r=0}^{n} a_r$.

(ii) Show that, if n is odd,

$$\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \tfrac{1}{2} b_{n+1}^2\,,$$

and give a corresponding result when n is even.

(iii) Show that, if n is even,

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 2\,,$$

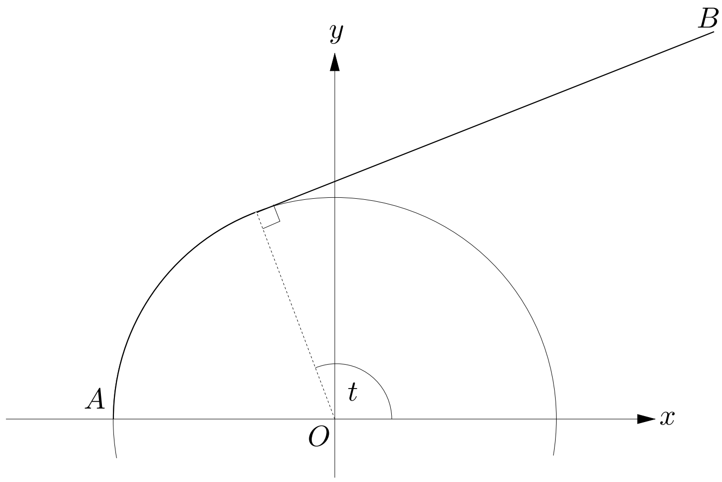
and give a corresponding result when n is odd.

8 The end A of an inextensible string AB of length π is attached to a point on the circumference of a fixed circle of unit radius and centre O. Initially the string is straight and tangent to the circle. The string is then wrapped round the circle until the end B comes into contact with the circle. The string remains taut during the motion, so that a section of the string is in contact with the circumference and the remaining section is straight.

Taking O to be the origin of cartesian coordinates with A at $(-1, 0)$ and B initially at $(-1, \pi)$, show that the curve described by B is given parametrically by

$$x = \cos t + t \sin t\,, \qquad y = \sin t - t\cos t\,,$$

where t is the angle shown in the diagram.



Find the value, t_0, of t for which x takes its maximum value on the curve, and sketch the curve.

Use the area integral $\displaystyle\int y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ to find the area between the curve and the x axis for $\pi \geqslant t \geqslant t_0$.

Find the area swept out by the string (that is, the area between the curve described by B and the semicircle shown in the diagram).

Section B: Mechanics

9 Two particles, A of mass $2m$ and B of mass m, are moving towards each other in a straight line on a smooth horizontal plane, with speeds $2u$ and u respectively. They collide directly. Given that the coefficient of restitution between the particles is e, where $0 < e \leqslant 1$, determine the speeds of the particles after the collision.

After the collision, B collides directly with a smooth vertical wall, rebounding and then colliding directly with A for a second time. The coefficient of restitution between B and the wall is f, where $0 < f \leqslant 1$. Show that the velocity of B after its second collision with A is

$$\tfrac{2}{3}(1-e^2)u - \tfrac{1}{3}(1-4e^2)fu$$

towards the wall and that B moves towards (not away from) the wall for all values of e and f.

10 A particle is projected from a point on a horizontal plane, at speed u and at an angle θ above the horizontal. Let H be the maximum height of the particle above the plane. Derive an expression for H in terms of u, g and θ.

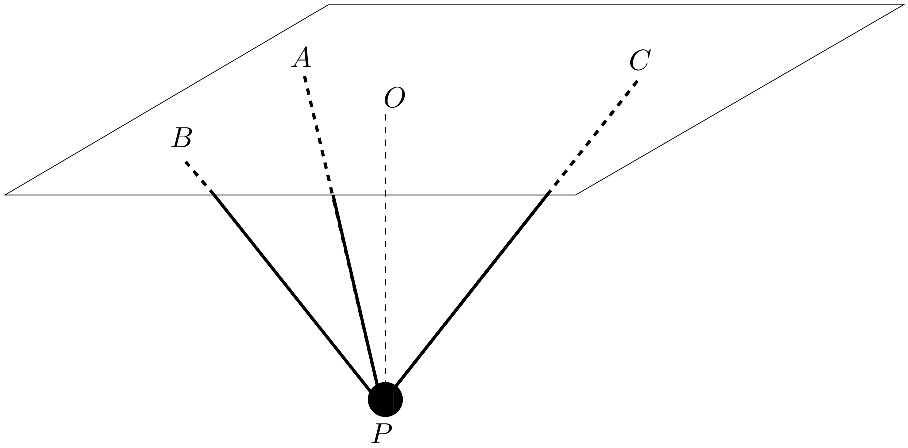
A particle P is projected from a point O on a smooth horizontal plane, at speed u and at an angle θ above the horizontal. At the same instant, a second particle R is projected horizontally from O in such a way that R is vertically below P in the ensuing motion. A light inextensible string of length $\tfrac{1}{2}H$ connects P and R. Show that the time that elapses before the string becomes taut is

$$(\sqrt{2}-1)\sqrt{H/g}\ .$$

When the string becomes taut, R leaves the plane, the string remaining taut. Given that P and R have equal masses, determine the total horizontal distance, D, travelled by R from the moment its motion begins to the moment it lands on the plane again, giving your answer in terms of u, g and θ.

Given that $D = H$, find the value of $\tan\theta$.

11 Three non-collinear points A, B and C lie in a horizontal ceiling. A particle P of weight W is suspended from this ceiling by means of three light inextensible strings AP, BP and CP, as shown in the diagram. The point O lies vertically above P in the ceiling.



The angles AOB and AOC are $90° + \theta$ and $90° + \phi$, respectively, where θ and ϕ are acute angles such that $\tan\theta = \sqrt{2}$ and $\tan\phi = \frac{1}{4}\sqrt{2}$.

The strings AP, BP and CP make angles $30°$, $90° - \theta$ and $60°$, respectively, with the vertical, and the tensions in these strings have magnitudes T, U and V respectively.

(i) Show that the unit vector in the direction PB can be written in the form

$$-\frac{1}{3}\,\mathbf{i} - \frac{\sqrt{2}}{3}\,\mathbf{j} + \frac{\sqrt{2}}{\sqrt{3}}\,\mathbf{k}\,,$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the usual mutually perpendicular unit vectors with $\mathbf{j}$ parallel to OA and $\mathbf{k}$ vertically upwards.

(ii) Find expressions in vector form for the forces acting on P.

(iii) Show that $U = \sqrt{6}V$ and find T, U and V in terms of W.

Section C: Probability and Statistics

12 Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points.

Xavier has probability p and Younis has probability $1-p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1-p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1-p^2}{2-p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him £1; if Younis wins the match, Xavier gives him £k. Find the value of k for which the game is fair in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

13 What property of a distribution is measured by its *skewness*?

(i) One measure of skewness, γ, is given by

$$\gamma = \frac{\mathrm{E}\big((X-\mu)^3\big)}{\sigma^3}\,,$$

where μ and σ^2 are the mean and variance of the random variable X. Show that

$$\gamma = \frac{\mathrm{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}\,.$$

The continuous random variable X has probability density function f where

$$\mathrm{f}(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Show that for this distribution $\gamma = -\dfrac{2\sqrt{2}}{5}$.

(ii) The *decile skewness*, D, of a distribution is defined by

$$D = \frac{\mathrm{F}^{-1}(\frac{9}{10}) - 2\mathrm{F}^{-1}(\frac{1}{2}) + \mathrm{F}^{-1}(\frac{1}{10})}{\mathrm{F}^{-1}(\frac{9}{10}) - \mathrm{F}^{-1}(\frac{1}{10})}\,,$$

where F^{-1} is the inverse of the cumulative distribution function. Show that, for the above distribution, $D = 2 - \sqrt{5}\,$.

The *Pearson skewness*, P, of a distribution is defined by

$$P = \frac{3(\mu - M)}{\sigma}\,,$$

where M is the median. Find P for the above distribution and show that $D > P > \gamma\,$.

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